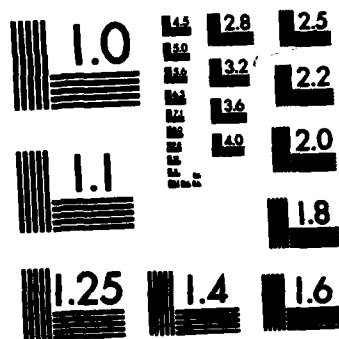MICROCOPY RESOLUTION TEST CHART

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# A COMPOSITE VELOCITY PROCEDURE FOR THE INCOMPRESSIBLE

# NAVIER-STOKES EQUATIONS

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## INTRODUCTION

One of the major differences between steady-state solution techniques for the Navier-Stokes equations and solution procedures for either inviscid potential or boundary layer problems is the treatment of the continuity equation. For inviscid flow, a potential function is determined entirely from continuity. The pressure is then obtained from the integrated momentum or Bernoulli equation. For boundary layers, the axial momentum and continuity equations determine the velocities. On the other hand, typical Navier-Stokes solvers, in effect, use continuity to obtain the density (or pressure), and the velocities result solely from the momentum equations. For large Reynolds number steady flows, it would appear that such a procedure is in marked conflict with both the asymptotic inviscid and boundary layer theories.

In the present paper, a boundary layer-relaxation procedure based on a new composite-velocity formulation for the incompressible Navier-Stokes system is described. The equations are interpreted and numerically approximated to reflect the composite nature of the flow. The procedure has also been developed independently for subsonic flow [1]. Unlike typical Navier-Stokes procedures that differ significantly from their incompressible flow counterparts, the present developments are essentially identical in both cases. Moreover, the extension to transonic flows should be direct.

The equations are written in a body-fitted orthogonal coordinate system so that arbitrary geometries can be treated. Application to internal and external flows are discussed. Specific geometries include a boattail simulator, the trailing edge of a plate, Joukowski airfoils and a curved channel.

In its final form, the present formulation has some features similar to the velocity-split technique due to Dodge [2]; however, this resemblance is only superficial. In the present analysis, a composite representation of inviscid and viscous region velocities is prescribed in the spirit of matched asymptotic expansions. The complete Navier-Stokes equations are solved. No simplifying approximations are required. The finite-difference form of the resulting equations are solved by a coupled strongly implicit procedure described previously by the authors [3].

## COMPOSITE FORMULATION

This formulation is designed for the calculation of large Reynolds number problems with a dominant flow direction, e.g. the $\xi$-direction. The gradients are

largest in surface normal ($\eta$) direction. The flow outside the thin viscous region is essentially inviscid and is represented by a potential function $\phi$; therefore, the following composite representation of the velocity field to reflect the matched asymptotic boundary layer-inviscid behavior is prescribed.

$$u = \frac{U}{h_1}(1 + \phi_\xi) = u_e U \quad , \qquad v = \frac{1}{h_2}\phi_\xi \tag{1}$$

Substitution of these expressions into the Navier-Stokes equations results, after some reorganization, in the following orthogonal system for $\phi$, $U$, $G$:

$$[h_3 U(1 + \phi_\xi)]_\xi + (h_3\phi_\eta)_\eta = 0 \qquad \text{(continuity)} \tag{2}$$

$$\frac{\partial u}{\partial t} + \frac{1}{h_1 h_2 h_3}\{[h_2 h_3 u_e^2 (u^2 - U)]_\xi + [h_3 h_1 u_e v(U-1)]_\eta\} + \frac{h_{1_\eta}}{h_1 h_2} u_e v(U-1) + \frac{u_e}{h_1}(U-1)u_{e_\xi}$$

$$= -\frac{1}{h_1}G_\xi - \frac{1}{h_1}\bar{G}'(\xi) + \text{viscous terms} \qquad (\xi\text{-momentum}) \tag{3}$$

$$G_\eta = -(U-1)[(\frac{u_e^2}{2})_\eta - \frac{h_{1_\eta}}{h_1}u_e^2 U] + \text{viscous terms} \qquad (\eta\text{-momentum}) \tag{4}$$

For internal flows $\bar{G}'(\xi)$ is determined from the global conservation of mass. In these equations

$$G = \frac{p}{\rho} + \frac{u_e^2 + v^2}{2} - \tilde{G}(\xi) \tag{5}$$

and is similar to the Bernoulli or total pressure for the inviscid region. However, $G$ is not assumed constant, but is determined by the calculation procedure. In an inviscid irrotational region, $U \to 1$ and continuity (2) reduces to the well known potential flow equation; the momentum equations (3, 4) are identically satisfied with $G$ = constant and $\bar{G}$ = 0. For internal flows, $G$ = 0 on one wall and $\bar{G}$ is determined by mass conservation. In the viscous region, the $\xi$-momentum equation determines $U$, while $\phi$ and, therefore, the pressure is obtained from the $\eta$-momentum equation. The two basic regions pertinent to large Reynolds number flows are appropriately described by this composite set of equations. This method of defining $v$ with a 'potential' was first tested for the flat plate boundary-layer ($U,\phi$) equations. The solution of the resulting 2-point boundary value problem reproduced the results obtained with standard methods based on the velocities $u$ and $v$. It should be pointed out that the present system of equations can be used for the solution of inviscid flows provided that Re = $\infty$ and $U$ = 1 is enforced at the solid surface. $G$ = 0 decouples the normal momentum equation from the axial momentum and continuity equations and represents the interacting boundary-layer approximation if axial diffusion is neglected. If, in addition, $\phi_\xi$ and $\phi_{\xi\xi}$ in equations (2) and (3) are replaced with their respective potential flow values, the usual boundary-layer approximation is recovered.

For the calculations of separated flows it is essential that U and $\phi_\xi$ are coupled in order to eliminate the pressure singularity. G which represents the Bernoulli constant in the inviscid flow can be treated explicitly. Finally, the $\eta$-momentum determines G. The necessary coupling between U and $\phi$ is attained with a coupled strongly implicit procedure for the solution of the algebraic system.

## BOUNDARY CONDITIONS

In the present formulation, the physical boundary conditions are represented directly by the following mathematical boundary conditions.

At a solid surface: $U = 0$; $\phi_\eta = 0$, and as $\eta \to \infty$, $\phi \to 0$, $U \to 1$.

For a finite body: $\phi$ or $\phi_\xi \to 0$ as $\xi \to \pm\infty$, $U(\xi,\eta) \to 1$ as $\xi \to -\infty$,

and $U_{\xi\xi}(\xi,\eta) \to 0$ as $\xi \to \infty$.

For bodies which are infinite in both directions, the inflow and outflow conditions are somewhat ambiguous. However, these conditions must be appropriately specified in order to obtain a meaningful solution. For the boattail geometry $\phi$ and U are prescribed at the inflow, while $U_{\xi\xi} \to 0$ and $\phi_\xi \to 0$ have been applied as outflow conditions. The $\phi_\xi$ condition, in the context of the composite formulation, is such as to eliminate the viscous-inviscid interaction. It should be noted that these boundary conditions are consistent with the mathematical character of the equations governing $\phi$ and U. For example no slip has been satisfied through the boundary-layer like variable U and not through $\phi_\xi$.

## SOLUTION PROCEDURE

The governing equations have been discretized using second-order accurate central-differencing for all $\phi$ derivatives. Central or boundary layer-like differencing has been used for U derivatives, except for the $(h_3 U)_\xi$ term in the continuity equation; this is backward differenced throughout. The resulting implicit algebraic system of equations has been solved iteratively using a coupled strongly implicit procedure (CSIP). The continuity and $\xi$-momentum equations for $\phi$ and U are solved in a coupled fashion, while the $\eta$-momentum equation for G is evaluated iteratively. In the $\xi$-momentum equation, G is treated as known during the iterations. Although G, the "inviscid" total pressure is evaluated explicitly from the $\eta$-momentum equation, the static pressure is unknown and depends upon the values of $\phi_\xi$ and $\phi_\eta$. Since $\phi$ is evaluated implicitly in the coupled algorithm, this implies that the pressure is also treated implicitly. This circumvents the separation singularity. Explicit artificial viscosity is not required for convergence and the CSIP allows for arbitrarily large values of $\Delta t$ once the effects of the inviscid initial conditions have been sufficiently smoothed.

## COUPLED 2 x 2 SOLUTION ALGORITHM

In an earlier paper, reference [3], the present authors have developed a coupled strongly implicit procedure for the stream function-vorticity form of the Navier-Stokes equations. This algorithm has the distinct advantage of being implicit in both the $\xi$ and $\eta$ directions, as well as allowing for the coupling of all the boundary conditions. It is this coupling, that eliminates the pressure singularity in the $(U, \phi)$ formulation. Furthermore, the method is unconditionally stable, allows for arbitrarily large $\Delta t$, converges faster than SOR, LSOR, ADI, etc., and is relatively insensitive to grid aspect ratio. The discretized version of the equations for $(U, \phi)$ can be written as:

$$(A + P) \, V^{n+1} = G + PV^n, \quad \text{where } V \text{ is the } (U, \phi) \text{ solution vector.}$$

P is chosen such that (A+P) can be decomposed into a lower and upper triangular form having a sparsity pattern similar to the original matrix A. This leads to a solution algorithm of the following form:

$$\begin{bmatrix} U_{ij} \\ \phi_{ij} \end{bmatrix}^{n+1} = \begin{bmatrix} GM_{1_{ij}} \\ GM_{2_{ij}} \end{bmatrix}^{n} + \begin{bmatrix} T_{1_{ij}} & T_{5_{ij}} \\ T_{3_{ij}} & T_{7_{ij}} \end{bmatrix}^{n} \begin{bmatrix} U_{i,j-1} \\ \phi_{i,j-1} \end{bmatrix}^{n+1} + \begin{bmatrix} T_{2_{ij}} & T_{4_{ij}} \\ T_{6_{ij}} & T_{8_{ij}} \end{bmatrix}^{n} \begin{bmatrix} U_{i-1,j} \\ \phi_{i-1,j} \end{bmatrix}^{n+1}$$

where n is the iteration index. Although the coupling accelerates the rate of convergence, it also increases the storage requirement by a factor of two. Considerable savings in storage can be realized by re-evaluating some of the coefficients $(T_{i,j})$ during the evaluation of U and $\phi$. However, this is achieved at moderate additional computational cost. In its present form the CSIP is slightly different from the one given in reference [3]. In the present case the forward and backward sweeps have been reversed in order to impart a certain degree of marching consistent with boundary layer procedures. As detailed in reference [3], the appropriate recurrence relationships can easily be obtained.

## RESULTS

Laminar flow solutions have been obtained for boattail simulator, Joukowski airfoil, finite plate and channel geometries. Reynolds numbers based on typical length scales ranged from $10^3$ to $10^5$. All the computations were started with arbitrary initial conditions. For the first 40 iterations $\Delta t = 1$, afterwhich $\Delta t$ was increased to $10^6$. Explicit artificial viscosity was not required for what were effectively steady-state calculations.

(a) <u>Boattail Simulator</u>: A typical streamline plot for an axisymmetric boattail geometry is shown in figure 1. The corner angle is approximately 30° and the body radius varies between 1 and 0.5. With 1800 grid points (60 x 30, 12 on the boattail), and Re = 7500, convergence to $10^{-3}$ is achieved in about 125 iterations. This takes about 10 minutes on the Amdahl 470/V6 or less than 1 minute on the Cray-1 computer.

(b) <u>Joukowski Airfoil</u>: A variety of airfoil thicknesses and Reynolds numbers have been considered. For low Reynolds numbers and small thickness ratio, the flow is unseparated. As either is increased, separation regions appear. Typical stream-line pattern and separation location for $t/c = 0.12$, $0.17$ and $Re = 10^3$ to $10^4$ are shown in fig. 2. As expected, the separation point moves upstream with increasing Reynolds number. This correlates well with solutions obtained with 2nd order boundary-layer theory [4]. Small changes in thickness ratio can lead to large variations on the recirculation region. The solutions are oscillatory for large t/c, Re.

(c) <u>Finite Flat Plate</u>: The flow past the trailing edge of a finite flat plate has been extensively investigated by triple-deck interacting boundary-layer global relaxation procedures [5]. In figure 3 the results of the present formulation are compared with some earlier computations. A 105 x 75 grid has been adequate to provide reasonable agreement.

(d) <u>Internal Flow</u>: Two dimensional straight and curved channels have been in-vestigated by the present technique. The curved channel was generated by using two streamlines of the boattail geometry. A variable $\xi$ and uniform-$\eta$ (60 x 30) grid was specified for this calculation. Uniform inflow conditions are prescribed. The entrance mass flow rate was 0.4066 and the Reynolds number based on the entrance channel width was Re = 2500. Global mass conservation was insured with the para-meter $\bar{G}'(\xi)$. Typical wall pressure distributions and velocity profiles are shown in figure 4.

## ACKNOWLEDGEMENT

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1. Khosla, P.K. and Rubin, S.G. (1982), AIAA Paper No. 82-0099.
2. Dodge, P.R. and Lieber, L.S. (1977), AIAA Paper No. 77-170.
3. Rubin, S.G. and Khosla, P.K. (1979), Comp. Fluids, Vol. 9, 2, p. 163.
4. Grossman, B. and Rubin, S.G., (1971), ZAMP, Vol. 22, 1, pp. 109-130.
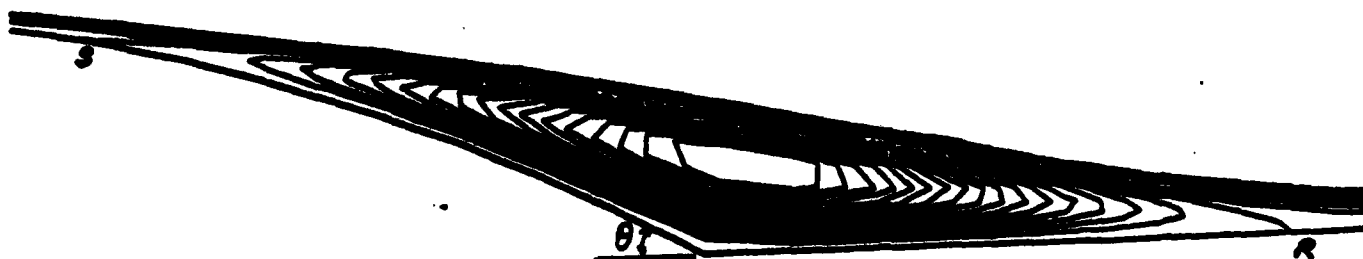5. Rubin, S.G., (1982), Von Karman Institute Lecture Notes, April 1982.

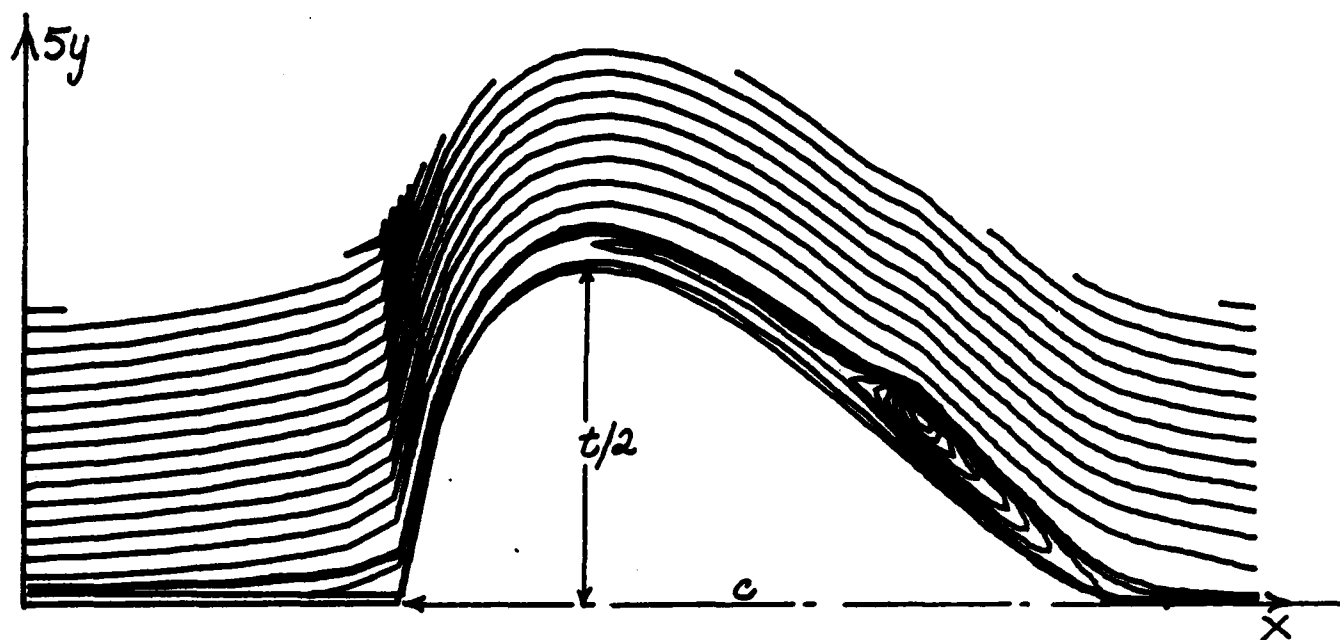Fig. 1. Streamline Contours for Boattail, $\theta$ = 30°, Re = 7500, 60 x 30 Grid.

Fig. 2a.   Streamline Contours for Joukowski Airfoil:   Re = 10,000, t/c = 0.12.

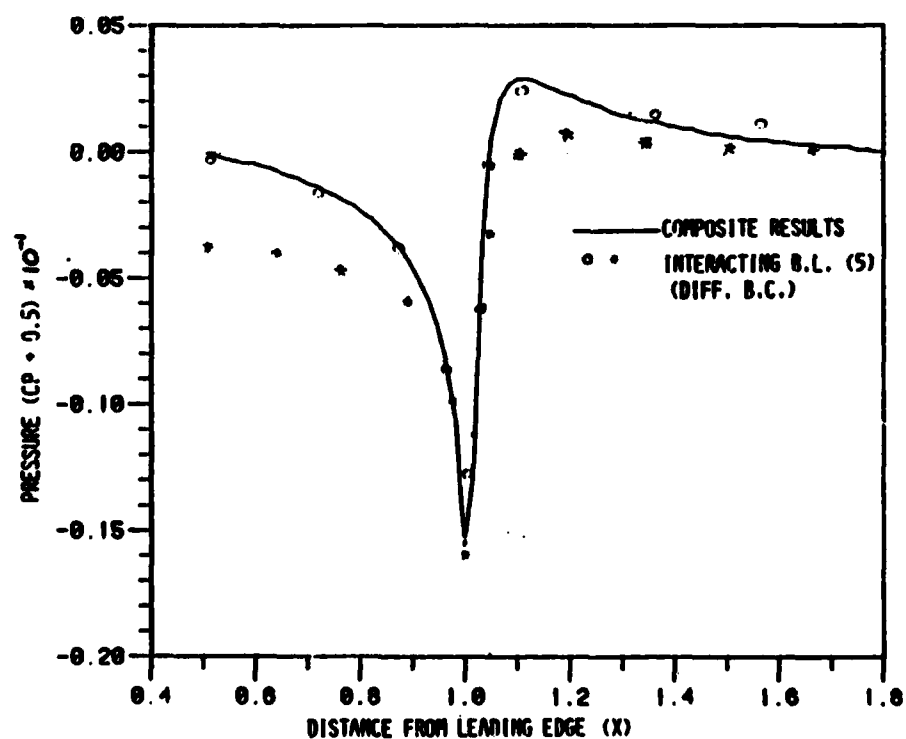| t/c \ Re | 1,000 | 5,000 | 7,500 | 10,000 | $10^5$ |
|---|---|---|---|---|---|
| 0.12 | No Separation | No Separation | 0.85 | 0.80 | Unsteady |
| 0.17 | No Separation | 0.47 | Unsteady | Unsteady | -- |

Fig. 2b.   Separation Location for Several Values of t/c, Re.



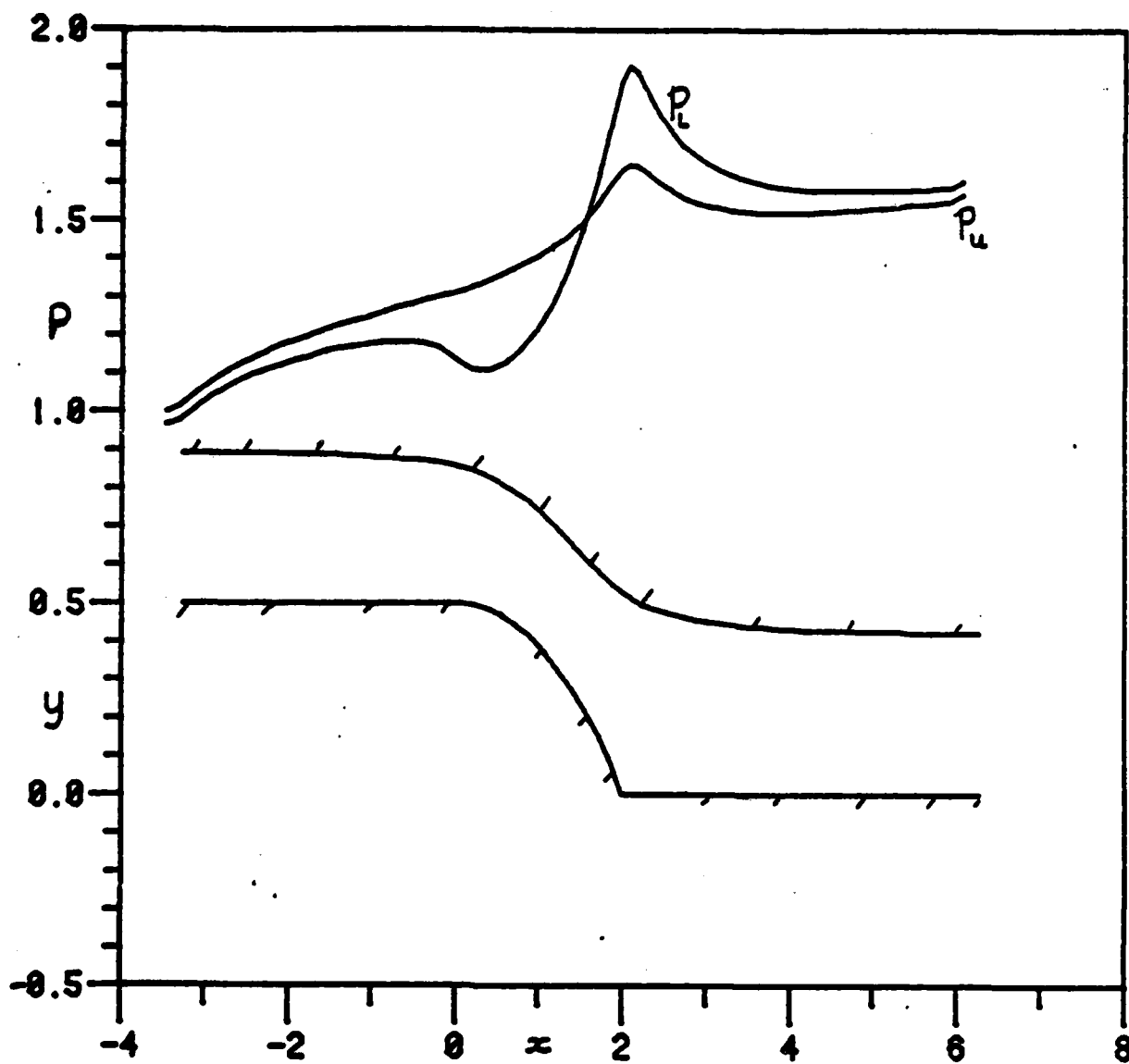Fig. 3.   Pressure Distribution Near Trailing Edge of Flat Plate:   Re = $10^5$.
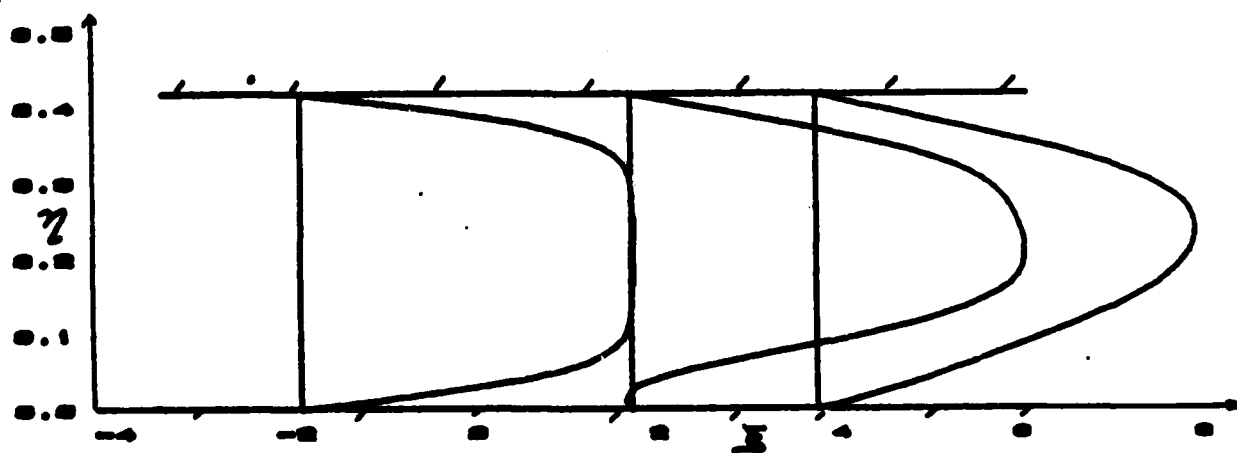
Fig. 4a. Wall Pressures and Channel Geometry.



Fig. 4b. Velocity Profiles in Channel-Transformed Plane.

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**20. ABSTRACT** *(Continue on reverse side if necessary and identify by block number)*

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